Linear Algebra II

April 10, 2019, Wednesday, 18:30 – 21:30, A. Jacobshal 01

Please write your name and student number on the exam ánd on the envelope. The exam contains 6 problems.

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1 $(5 + 5 + 5 = 15 \text{ pts})$ **Inner product spaces**

(a) For a given a complex vector space $\mathcal{V}$, give the definition of complex inner product $\langle x, y\rangle$ on $\mathcal{V}$.

(b) Now consider $\mathcal{V} = \mathbb{C}^n$ with $\langle x, y\rangle := y^H x$. Prove that this is an inner product on $\mathbb{C}^n$.

(c) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show that $\langle x, Ax\rangle$ is real for all $x \in \mathbb{C}^n$.

2 $(5 + 10 = 15 \text{ pts})$ **Cayley-Hamilton**

Consider the real matrix

$$A = \begin{bmatrix} 9 & -6 \\ 5 & -3 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Compute α and β such that $\alpha A^5 + \beta A = I$.

3 $(4 + 4 + 3 + 4 = 15 \text{ pts})$ **Positive definite matrices**

Let $A \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix.

(a) Prove that all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are real and positive.

(b) Prove that $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the eigenvalues of A^2

(c) Show that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the singular values of A.

(d) Let Q be an orthogonal matrix such that $Q^T AQ = \Lambda$ with Λ the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal. Determine a singular value decomposion of A in terms of the eigenvalues λ_i and the matrix Q.

4 (6 + 6 + 3 = 15 pts) **Singular value decomposition**

Consider the 3 × 2 matrix

$$M = \begin{bmatrix} 3 & 3 \\ -\sqrt{2} & \sqrt{2} \\ 3 & 3 \end{bmatrix}.$$

(a) Compute the singular values of M

(b) Find a singular value decomposition for M.

(c) Find the best rank 1 approximation of M.

5 (4 + 4 + 7 = 15 pts) **Characteristic polynomial and minimal polynomial**

Let A be a complex $n \times n$ matrix.

(a) Give the definition of minimal polynomial $p_{\min}(z)$ of the matrix A.

(b) Let λ be an eigenvalue of A with eigenvector x. Let $p(z)$ be a nonzero polynomial. Show that $p(A)x = p(\lambda)x$.

(c) Prove that if λ is an eigenvalue of A then $p_{\min}(\lambda) = 0$.

6 (3 + 4 + 4 + 4 = 15 pts) **Jordan canonical form**

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A.

(b) Compute the minimal polynomial of A.

(c) Determine the Jordan canonical form J of A.

(d) Compute a nonsingular matrix S such that $S^{-1}AS = J$.

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10 pts free